

Hopf algebraic structure of the parabosonic and parafermionic algebras and paraparticle generalization of the Jordan Schwinger map

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Abstract: The aim of this paper is to show that there is a Hopf structure of the parabosonic and parafermionic algebras and this Hopf structure can generate the well known Hopf algebraic structure of the Lie algebras, through a realization of Lie algebras using the parabosonic (and parafermionic) extension of the Jordan Schwinger map. The differences between the Hopf algebraic and the graded Hopf superalgebraic structure on the parabosonic algebra are discussed.

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I. Introduction

Jordan at 1935 proposed a realization of Lie algebras using boson or fermion creation - destruction operators. This realization was rediscovered by Schwinger at 1953, see the discussion of the Jordan - Schwinger map in [1][Chap. 5] and in [2][§2.3]. The realization of a Lie algebra by bosons corresponds to the symmetric representation of the Lie algebra, while the realization by using fermions corresponds to the antisymmetric representation of the Lie algebra. The case of the $u(N)$ algebra was presented in [3]. The bosons and the fermions are special cases of parabosons and parafermions, which are introduced by Green [4]. The parabosonic (and correspondingly the parafermionic) algebra is a generalization of the usual bosonic (fermionic) algebra leading to generalized alternatives of the Bose - Einstein (Fermi-Dirac) statistics or to field theories based on paraparticles, all the related bibliography and details can be found in [5]. The N parabosons were used for constructing realization of a $sp(2N)$ algebra, this construction is based on the idea of using parabosons, rather than usual bosons. The same idea can be applied in the case of the $so(2N)$ algebra but N parafermions are used [5][§3.2]. Biswas and Soni [6] used systematically parabosons or parafermions for a Jordan -Schwinger realization of a $u(N)$ algebra. In the same paper realizations of the $so(2N + 1)$ or $sp(2N)$ algebras, using parafermions and parabosons, were discussed in a similar way as in [5, 7, 8]. Also a realization of the $g(M/N)$ super algebra is proposed by using M parafermionic and N parabosonic operators, by extending the corresponding realizations based on the use of usual fermions and bosons. Palev[9] has shown that, the bilinear combinations of the paraoperators yield the superalgebra $gl(n/m)$ (see also in the same paper the realizations of $so(2n+1)$ and of $osp(1/2m)$). Later the same author [10] has also proved that the parabosonic and parafermionic algebras can be used for constructing realizations of $osp(2N + 1/2M)$ algebras.

The extension of the Jordan - Schwinger map as a method of a realization of every Lie algebra, using parabosons and parafermions was originally published since 1971 in a local journal by Palev[11]. This work is not widely known, even we ignored it, when the first version of this work was printed as a preprint.

The fact that Lie algebras and superalgebras have a Hopf algebra structure, constitutes a strong indication that the parabosonic and parafermionic algebras might possess a Hopf algebraic structure too. If this is true, then the Hopf algebra structure of the Lie algebra should be consistent with the supposed Hopf algebra structure of the parabosonic and parafermionic al-

gebra. Another strong indication of the existence of the Hopf structure in the parabosonic algebra can be conjectured by the recent investigation by Macfarlane[12], which has proved that the one dimensional Calogero-Vassiliev oscillator algebra is closely related to the one dimensional parabosonic algebra. In [13] and [14] a Hopf algebra structure was proposed for the one dimensional Calogero-Vassiliev oscillator algebra. Therefore a natural idea is to transfer the Hopf structure of the Calogero-Vassiliev algebra to the case of the parabosonic algebra. In this paper we show that both the parabosonic and parafermionic algebras admit a Hopf algebra structure. Also the Jordan-Schwinger map is a Hopf homomorphism from the Lie algebras to the parafermionic or parabosonic algebras. That means that the rules of the Hopf algebra structure of the Lie algebra can be deduced from the rules of the Hopf algebra structure of the parabosonic or parafermionic algebra. Also we show that for an Lie algebra of N generators which is realized using N pairs of creation - destruction parabosonic (or parafermionic) operators, then the N parabosonic (parafermionic) destruction operators are the components of a covariant tensor algebra, while the N creation operators are the contravariant components of a tensor corresponding to the adjoint representation of the Lie algebra.

Ganchev and Palev [15] have shown that there is a realization of the Lie superalgebra $\text{osp}(1, 2n)$ by using the parabosonic algebra. In this realization, the parabosonic generators are the odd generators of the $\text{osp}(1, 2n)$. The universal $U(\text{osp}(1, 2n))$ algebra is a \mathbb{Z}_2 graded associative algebra, which has a usual Hopf superalgebra structure [16]. This structure is closely related to the \mathbb{Z}_2 grading of the $\text{osp}(1, 2n)$ algebra. In this paper we show that the Parabosonic algebra is also a Hopf algebra, this structure is obtained by adding a generator to the parabosonic algebra. Therefore the parabosonic algebra can be regarded either as a Hopf superalgebra either as a Hopf algebra. The differences of these algebra structures are examined in section II. The above cited peculiarity of the parabosonic algebra is not valid for the case of parafermionic algebras.

II. Parabosonic realization of a Lie Algebra

Let consider a finite dimensional Lie algebra \mathcal{L} , generated by the generators X_1, X_2, \dots, X_n and the commutation relations:

$$[X_i, X_j] = \sum_{k=1}^n c_{ij}^k X_k \quad (1)$$

The universal enveloping algebra $U(\mathcal{L})$ is a Hopf algebra with commultipli-

cation, co-unit and antipode, which are defined by the well known relations:

$$\begin{aligned}\Delta(X_i) &= X_i \otimes 1 + 1 \otimes X_i \\ \epsilon(X_i) &= 0 \\ S(X_i) &= -X_i\end{aligned}\tag{2}$$

Let consider the parabosonic algebra $\mathcal{P}(n)$, which is the associative algebra generated by n generators b_1, b_2, \dots, b_n satisfying the trilinear commutation relations:

$$\begin{aligned}\left[b_k, \{b_\ell^\dagger, b_m\} \right] &= 2\delta_{k\ell} b_m \\ \left[b_k, \{b_\ell^\dagger, b_m^\dagger\} \right] &= 2\delta_{k\ell} b_m^\dagger + 2\delta_{km} b_\ell^\dagger \\ \left[b_k, \{b_\ell, b_m\} \right] &= 0\end{aligned}\tag{3}$$

The generators of the parabosonic algebra define a $u(n)$ algebra by putting

$$N_{\ell m}^{[B]} = \frac{1}{2} \{b_\ell^\dagger, b_m\}\tag{4}$$

The trilinear equations (3) imply that the operators $N_{\ell m}^{[B]}$ are generators of a $u(n)$ algebra, because the following commutation relations are satisfied:

$$\left[N_{k\ell}^{[B]}, N_{mn}^{[B]} \right] = \delta_{\ell m} N_{kn}^{[B]} - \delta_{kn} N_{m\ell}^{[B]}\tag{5}$$

The linear Casimir operator of the $u(n)$ algebra is defined by:

$$N^{[B]} = \sum_{i=1}^n N_{ii}^{[B]} = \frac{1}{2} \sum_{i=1}^n \{b_i^\dagger, b_i\}\tag{6}$$

and it satisfies the commutation relations:

$$\left[N^{[B]}, b_i^\dagger \right] = b_i^\dagger, \quad \left[N^{[B]}, b_i \right] = -b_i\tag{7}$$

Starting from the $N^{[B]}$ operator we can define the operator

$$K = \exp[i\pi N^{[B]}], \quad K^\dagger = K^{-1} = \exp[-i\pi N^{[B]}\tag{8}$$

The above commutation relations imply

$$\begin{aligned}KK^\dagger &= K^\dagger K = 1 \\ \{K, b_i\} &= \{K, b_i^\dagger\} = \{K^\dagger, b_i\} = \{K^\dagger, b_i^\dagger\} = 0\end{aligned}\tag{9}$$

The parabosonic algebra $\mathcal{P}(n)$ admits a Hopf algebra structure, with a coproduct Δ_p , co-unit ϵ_p and antipode S_p given by:

$$\begin{aligned}\Delta_p(b_i) &= b_i \otimes 1 + K \otimes b_i \\ \Delta_p(b_i^\dagger) &= b_i^\dagger \otimes 1 + K^\dagger \otimes b_i^\dagger \\ \Delta_p(K) &= K \otimes K \quad \text{and} \quad \Delta_p(K^\dagger) = K^\dagger \otimes K^\dagger \\ \epsilon_p(b_i) &= \epsilon_p(b_i^\dagger) = 0, \quad \epsilon_p(K) = \epsilon_p(K^\dagger) = 1 \\ S_p(b_i) &= b_i K^\dagger, \quad S_p(b_i^\dagger) = b_i^\dagger K, \quad S_p(K) = K^\dagger, \quad S_p(K^\dagger) = K\end{aligned}\tag{10}$$

The above defined coproduct Δ_p is an algebra homomorphism from the space $\mathcal{P}(n)$ into the tensor product $\mathcal{P}(n) \otimes \mathcal{P}(n)$.

$$\mathcal{P}(n) \xrightarrow{\Delta_p} \mathcal{P}(n) \otimes \mathcal{P}(n)$$

The tensor product $\mathcal{P}(n) \otimes \mathcal{P}(n)$ has the usual tensor algebra structure, i.e. there is a product defined as follows:

$$(a \otimes b) \cdot (c \otimes d) = ac \otimes bd\tag{11}$$

It is not difficult to show that the above definitions do indeed satisfy the well known consistency conditions

$$\begin{aligned}(id \otimes \Delta)\Delta(a) &= (\Delta \otimes id)\Delta(a) \\ (id \otimes \epsilon)\Delta(a) &= (\epsilon \otimes id)\Delta(a) = a \\ m(id \otimes S)\Delta(a) &= m(S \otimes id)\Delta(a) = \epsilon(a)I\end{aligned}\tag{12}$$

Using the above defined generators a realization of the Lie algebra \mathcal{L} can be defined on the parabosonic algebra $\mathcal{P}(n)$ by using a Jordan - Schwinger map[2]:

$$\mathcal{L} \ni X_i \xrightarrow{\mathcal{J}} \mathcal{J}(X_i) \in \mathcal{P}(n)$$

where

$$\mathcal{J}(X_i) = \sum_{k\ell} c_{ki}^\ell N_{k\ell}^{[B]} = \frac{1}{2} \sum_{k\ell} c_{ki}^\ell \{b_k^\dagger, b_\ell\}\tag{13}$$

After trivial calculations one can verify that the generators $\mathcal{J}(X_i)$ satisfy the Lie algebra commutation relations (1), because the structure constants of the Lie algebra satisfy the Jacobi equality:

$$c_{ij}^p c_{pk}^m + c_{jk}^p c_{pi}^m + c_{ki}^p c_{pj}^m = 0$$

The Jordan - Schwinger map can be extended to a map from the universal enveloping algebra $U(\mathcal{L})$ to the parabosonic algebra $\mathcal{P}(n)$. This extension of the Jordan - Schwinger map, was initially introduced by Palec[11] since 1971. Unfortunately, it was published in a local journal, and since its publication, it is not widely known.

The existence of the Hopf algebraic structure of the parabosonic algebra $\mathcal{P}(n)$, given by equation (10), and the definition of the operators $N_{ij}^{[B]}$, see equation (4) imply that

$$\begin{aligned}\Delta_p(N_{ij}^{[B]}) &= N_{ij}^{[B]} \otimes 1 + 1 \otimes N_{ij}^{[B]} \\ \epsilon_p(N_{ij}^{[B]}) &= 0 \\ S_p(N_{ij}^{[B]}) &= -N_{ij}^{[B]}\end{aligned}$$

These relations can be shown by lengthy but trivial algebraic calculations. For clarification reasons we reproduce here the proof of the first one of these relations.

$$\begin{aligned}\Delta_p(N_{ij}^{[B]}) &= \frac{1}{2} \Delta_p(\{b_i^\dagger, b_j\}) = \frac{1}{2} \{\Delta_p(b_i^\dagger), \Delta_p(b_j)\} = \\ &= \frac{1}{2} \{b_i^\dagger \otimes 1 + K^\dagger \otimes b_i^\dagger, b_j \otimes 1 + K \otimes b_j\} = \\ &= \frac{1}{2} \left((b_i^\dagger \otimes 1 + K^\dagger \otimes b_i^\dagger) (b_j \otimes 1 + K \otimes b_j) + \right. \\ &\quad \left. + (b_j \otimes 1 + K \otimes b_j) (b_i^\dagger \otimes 1 + K^\dagger \otimes b_i^\dagger) \right) = \\ &= \frac{1}{2} \left(b_i^\dagger b_j \otimes 1 + b_i^\dagger K \otimes b_j + K^\dagger b_j \otimes b_i^\dagger + K^\dagger K \otimes b_i^\dagger b_j + \right. \\ &\quad \left. + b_j b_i^\dagger \otimes 1 + K b_i^\dagger \otimes b_j + b_j K^\dagger \otimes b_i^\dagger + K K^\dagger \otimes b_j b_i^\dagger \right) = \\ &= \frac{1}{2} \left(\{b_i^\dagger, b_j\} \otimes 1 + \{b_i^\dagger, K\} \otimes b_j + \{K^\dagger, b_j\} \otimes b_i^\dagger + 1 \otimes \{b_i^\dagger, b_j\} \right) = \\ &= N_{ij}^{[B]} \otimes 1 + 1 \otimes N_{ij}^{[B]}\end{aligned}$$

By using the parabosonic realization (13) of the Lie algebra \mathcal{L} , the familiar Hopf algebra relations are satisfied:

$$\begin{aligned}\Delta_p(\mathcal{J}(X_i)) &= \mathcal{J}(X_i) \otimes 1 + 1 \otimes \mathcal{J}(X_i) = (\mathcal{J} \otimes \mathcal{J}) \circ \Delta(X_i) \\ \epsilon_p(\mathcal{J}(X_i)) &= 0 = \epsilon(X_i) \\ S_p(\mathcal{J}(X_i)) &= -\mathcal{J}(X_i) = \mathcal{J}(S(X_i))\end{aligned}$$

Therefore we have shown that the trilinear parabosonic definition (3) and the "deformed" -like Hopf structure (10) of the parabosonic algebra imply the Hopf structure of the $u(n)$ algebra.

A direct implication of the above relations is that, the Hopf algebraic structure of the enveloping algebra $U(\mathcal{L})$ can be deduced from the Hopf algebraic structure of the parabosonic algebra $\mathcal{P}(n)$.

The following diagrams are commutative:

$$\begin{array}{ccc} U(\mathcal{L}) & \xrightarrow{\Delta} & U(\mathcal{L}) \otimes U(\mathcal{L}) \\ \mathcal{J} \downarrow & & \downarrow \mathcal{J} \otimes \mathcal{J} \end{array} \quad (14)$$

$$\begin{array}{ccc} \mathcal{P}(n) & \xrightarrow{\Delta_p} & \mathcal{P}(n) \otimes \mathcal{P}(n) \\ U(\mathcal{L}) & \xrightarrow{\epsilon} & C \\ \mathcal{J} \downarrow & & \downarrow \text{Id} \\ \mathcal{P}(n) & \xrightarrow{\epsilon_p} & C \end{array} \quad (15)$$

and

$$\begin{array}{ccc} U(\mathcal{L}) & \xrightarrow{S} & U(\mathcal{L}) \\ \mathcal{J} \downarrow & & \downarrow \mathcal{J} \\ \mathcal{P}(n) & \xrightarrow{S_p} & \mathcal{P}(n) \end{array} \quad (16)$$

The above diagrams prove that, the Jordan map \mathcal{J} is a Hopf algebra homomorphism[17]

$$\begin{aligned} \Delta_p \circ \mathcal{J} &= (\mathcal{J} \otimes \mathcal{J}) \circ \Delta \\ \epsilon_p \circ \mathcal{J} &= \epsilon \\ S_p \circ \mathcal{J} &= \mathcal{J} \circ S \end{aligned}$$

The realization of any Lie algebra by using boson or fermions operators was initially discovered by Jordan and later was rediscovered by Schwinger of the $su(2)$ case, see [2][§2.4]. In this construction the formula (13) is used, but the b_i 's correspond to usual bosons. In this paper, we have extended the notion of the "Jordan map" from the boson (or fermion) case to the paraboson case, i.e. and we have proved that:

Proposition 1: *The extended Jordan map, which is defined by (13), is a Hopf algebra homomorphism from the Hopf algebra $U(\mathcal{L})$ into the parabosonic*

Hopf algebra $\mathcal{P}(n)$

$$U(\mathcal{L}) \xrightarrow{\mathcal{J}} \mathcal{P}(n)$$

Another trivial result is:

Proposition 2: *The set of n parabosonic destruction (or creation) operators $\{b_i\}$, $i = 1, \dots, n$ (or $\{b_i^\dagger\}$, $i = 1, \dots, n$) are adjoint tensor covariant (correspondingly contravariant) operators of the Lie algebra \mathcal{L}*

That is true because the definition (13) of the parabosonic realization of the Lie algebra generators and the trilinear commutation relations for the parabosons (3) imply the relations:

$$\begin{aligned} [\mathcal{J}(X_i), b_j] &= \sum_{k=1}^n c_{ji}^k b_k \\ [\mathcal{J}(X_i), b_j^\dagger] &= - \sum_{k=1}^n c_{ik}^j b_k^\dagger \end{aligned} \quad (17)$$

and the structure constants c_{ij}^k are the matrix elements of the adjoint representation of the Lie generators X_i .

It should be noted that a Hopf superalgebra structure can be obtained by using the results of ref. [15], where the parabosonic algebra $\mathcal{P}(n)$ is formulated as a graded \mathbb{Z}_2 superalgebra with the parabosonic annihilation and creation operators being graded as odd elements of the $\text{osp}(1, 2n)$ Lie superalgebra. In this structure the parabosonic algebra $\mathcal{P}(n)$ is mapped in the universal enveloping algebra $U(\text{osp}(1, 2n))$. The graded algebra $U(\text{osp}(1, 2n))$ is a Hopf superalgebra [16], that means that there is a superalgebra coproduct Δ_{gr} , a counit ϵ and a superalgebra antipode S_{gr} , which have the usual Hopf algebra properties (12). These operators can be defined on the parabosonic algebra $\mathcal{P}(n)$. The coproduct Δ_{gr} is a mapping from the algebra $\mathcal{P}(n)$ into the graded tensor superalgebra $\mathcal{P}(n) \otimes_{gr} \mathcal{P}(n)$. The algebra $\mathcal{P}(n) \otimes_{gr} \mathcal{P}(n)$ is the usual tensor algebra with a graded multiplication, namely:

$$(a \otimes b) \cdot (c \otimes d) = (-1)^{\deg(b)\deg(c)} ac \otimes bd \quad (18)$$

and

$$\begin{aligned} \Delta_{gr}(a) &= a \otimes 1 + 1 \otimes a \\ \epsilon(a) &= 0, \quad \epsilon(1) = 1 \\ S_{gr}(a) &= -a, \quad S_{gr}(ab) = (-1)^{\deg(a)\deg(b)} S_{gr}(b) S_{gr}(a) \end{aligned} \quad (19)$$

Therefore the parabosonic $\mathcal{P}(n)$ is a Hopf superalgebra [16], also it is a Hopf algebra as we have shown in this section by adding an additional generator K which is defined by equation (8).

III. Parafermionic realization of a Lie algebra

Another similar construction can be achieved by considering the parafermionic algebra $\mathcal{F}(n)$, which is generated by the elements f_1, f_2, \dots, f_n , which satisfy the trilinear commutation relations:

$$\begin{aligned} [f_k, [f_\ell^\dagger, f_m]] &= 2\delta_{k\ell}f_m \\ [f_k, [f_\ell^\dagger, f_m^\dagger]] &= 2\delta_{k\ell}f_m^\dagger - 2\delta_{km}f_\ell^\dagger \\ [f_k, [f_\ell, f_m]] &= 0 \end{aligned} \quad (20)$$

This algebra has a simple Hopf algebraic structure given by:

$$\begin{aligned} \Delta_f(f_i) &= f_i \otimes 1 + 1 \otimes f_i \\ \Delta_f(f_i^\dagger) &= f_i^\dagger \otimes 1 + 1 \otimes f_i^\dagger \\ \epsilon_f(f_i) &= \epsilon(f_i^\dagger) = 0 \\ S_f(f_i) &= -f_i, \quad S_f(f_i^\dagger) = -f_i^\dagger \end{aligned} \quad (21)$$

and can be easily checked that the consistency relations (12) are satisfied. $u(n)$ generators can be defined, similarly as it was given in equation (4):

$$N_{\ell m}^{[F]} = \frac{1}{2} [f_\ell^\dagger, f_m], \quad N^{[F]} = \sum_{i=1}^n N_{ii}^{[F]} \quad (22)$$

the Lie algebra can be realized by using a Jordan - Schwinger map:

$$\mathcal{J}(X_i) = \sum_{k\ell} c_{ki}^\ell N_{k\ell}^{[F]} = \frac{1}{2} \sum_{k\ell} c_{ki}^\ell [f_k^\dagger, f_\ell] \quad (23)$$

The generalization of the Propositions 1 and 2 in the case of parafermions is straightforward:

Proposition 1a: *The extended Jordan map, which is defined by (23), is a Hopf algebra homomorphism from the Hopf algebra $U(\mathcal{L})$ into the parafermionic Hopf algebra $\mathcal{F}(n)$*

$$U(\mathcal{L}) \xrightarrow{\mathcal{J}} \mathcal{F}(n)$$

and

Proposition 2a: *The set of n parafermionic destruction (or creation) operators $\{f_i\}$, $i = 1, \dots, n$ (or $\{f_i^\dagger\}$, $i = 1, \dots, n$) are adjoint tensor covariant (correspondingly contravariant) operators of the Lie algebra \mathcal{L}*

IV. Discussion

The realizations of the Lie algebras, by using a generalized Jordan map seems to be a useful tool. It is well known the importance of the Jordan -Schwinger map [1] in the study of the representations of the Lie Algebras. The proposed extension of this fundamental map in the case of the parabosonic and parafermionic algebra, opens several problems to be solved:

1. *The relation of the parabosonic and parafermionic realizations to the known representations of the Lie algebras.*

It is well known that the usual bosonic (or fermionic) Jordan map leads to the symmetric (antisymmetric) unitary representations of the Lie algebras. The representations of the parabosonic and parafermionic algebras are characterized by a positive integer p and a vacuum state $|0\rangle$:

$$b_i b_j^\dagger |0\rangle = p \delta_{ij} |0\rangle$$

Starting from this representation one can construct a representation of the Lie algebra. The connection of this representation to the known representations is not yet known. The proposed parabosonic extension of the Jordan - Schwinger map uses the adjoint representation of the Lie algebra. A more general extension of this construction can be defined by using the other representations of the Lie algebra.

2. *The construction of the dual Hopf algebra corresponding to the Hopf algebra of parafermions and parabosons.* The dual Hopf algebra of the universal enveloping algebra $U(\mathcal{L})$ is the set of the smooth real functions $C^\infty(G)$ defined on the local Lie group corresponding to the Lie algebra \mathcal{L} . The structure of the dual Hopf algebra of the parabosonic or the parafermionic algebra is not known.
3. *The q -deformed versions of parabosonic or the parafermionic algebras* Another open problem is the q -deformed extension of the Jordan-Schwinger map. It is well known that the ordinary one dimensional q -deformed bosons leads to the Jordan -Schwinger map in the special case of the $su_q(2)$, and analogous constructions are known for the $u_q(N)$

algebras. As far as we know, there is not any generalization of the Jordan - Schwinger map for the q -deformed versions of the parabosonic or parafermionic algebra, while descriptions of the $\text{osp}_q(1/m)$ [18] and $\text{osp}_q(2n + 1/m)$ have been recently studied [19] by using q -deformed generalizations of parabosonic and parafermionic algebras. A very interesting problem seems to be the investigation of the q -deformed generalizations of the parabosonic or parafermionic algebras and their relation with the Hopf algebraic structure of the quantum groups.

Beyond the importance of the study of representations of the generalized Jordan - Schwinger map, we have shown that this map is a Hopf algebra homomorphism. This fact implies that, the Hopf algebraic structure of the universal enveloping algebra of a Lie algebra is generated by the Hopf algebraic structure of the parabosonic and parafermionic algebras. From this point of view, the Hopf structure of parabosonic and parafermionic algebras is more fundamental, than the Hopf structure of Lie algebras. The extension of these ideas in the case of superalgebras is under investigation.

We must also notify, that Greenberg[20] conjectured, that the quarks are parafermions of order $p = 3$. The basic symmetry group of the naive quark theory is the $\text{SU}(3)$, therefore the Greenberg's assumption is related to the proposed realization of the $\text{su}(3)$ algebra by using eight parafermions of order $p = 3$.

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